International Journal of Scientific & Engineering Research Volume 4, Issue 9, January-2013
ISSN 2229-5518

Jacobian integral and Stability of the equilibrium position of the centre of mass of an extensible cable connected satellites system in the elliptic orbit

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Abstract:

During the motion of a cable connected satellites system, at least one equilibrium point exists when perturbative forces like Air-resistance, solar pressure, shadow of the earth due to solar pressure, magnetic force and oblateness of the earth act simultaneously. We have obtained two equilibrium points in case of perturbative forces like shadow of the earth due to solar pressure and oblateness of the earth acting together on the motion of two extensible cable-connected satellites. Liapunov's theorem has been used to examine the stability of the equilibrium points

1.Introduction

The present paper is devoted to examine the stability of the equilibrium points of the centre of mass of a system of two satellites connected by a light, flexible and extensible string under the influence of shadow of the oblate earth due to solar pressure in elliptic orbit. Beletsky; V.V is the pioneer worker in this field. It is the generalisation of the works done by Beletsky; V.V, Singh; R.B, Sinha; S.K, and Singh; C.P,

2.Equation of motion

The equation of motion of one of the two satellites moving along a keplerian elliptic orbit in Nechvill's coordinates can be obtained by exploiting Lagrange's equation of motion of first kind in the form for two dimensional case.

$$x''-2y'-3x\rho-\frac{4Bx}{\rho}+A\Psi_1\rho^3\cos\in\cos(\nu-\alpha)=-\overline{\lambda_\alpha}\rho^4\left[1-\frac{l_0}{\rho r}\right]x$$

$$y''+2x'+\frac{By}{\rho}-A\Psi_1\rho^3\cos\in\sin(\nu-\alpha)=-\overline{\lambda_\alpha}\rho^4\left[1-\frac{l_0}{\rho r}\right]y \qquad (1)$$

where, $\rho = \frac{1}{1+e\cos\nu}$; $r=\sqrt{x^2+y^2}$

v = True anomaly of the centre of mass of the system

$A = \frac{P^3}{\mu}\left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right)$ = Solar pressure parameter

$B = \frac{3k_2}{P^2}$ = Oblateness force parameter.

Ψ_1 = Shadow function parameter.

$$\bar{\lambda}_\alpha = \frac{p^3}{\mu}\frac{\lambda}{\ell_0}\frac{(m_1+m_2)}{m_1 m_2}; \ell_0 = Natural\ length\ of\ the\ string$$

$$\lambda = \text{mod}\,ulus\ of\ elasticity$$

Here dashes denote the differentiation with respect to true anomaly v

The condition of constraint is given by

$$x^2 + y^2 \le \frac{\ell_0^2}{\rho^2} \tag{2}$$

To find the Jacobian integral of the problem, the averaged values of the secular terms due to periodic terms presents in the equations of motion (1) can be deduced as given follow :

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\rho}\,dv = 1;\ \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\rho^2}\,dv = \left(1+\frac{e^2}{2}\right)$$

$$\frac{1}{2\pi}\int_0^{2\pi}\rho\,dv = \frac{1}{(1-e^2)^{\frac{1}{2}}};\ \frac{1}{2\pi}\int_0^{2\pi}\rho^3\,dv = \frac{2+e^2}{2(1-e^2)^{\frac{5}{2}}};\ \frac{1}{2\pi}\int_0^{2\pi}\rho^4\,dv = \frac{2+3e^2}{2(1-e^2)^{\frac{7}{2}}};$$

$$\frac{1}{2\pi}\int_0^{2\pi}A\rho^3\psi_1\cos\in\cos(v-\alpha)dv = \frac{1}{2\pi}\left[\int_{\substack{-\theta\\ \Psi_1=0}}^{\theta}A\rho^3\psi_1\cos\in\cos(v-\alpha)dv + \int_{\substack{\theta\\ \Psi_1=1}}^{2\pi}A\rho^3\psi_1\cos\in\cos(v-\alpha)dv\right]$$

$$= \frac{-(2+e^2)}{2(1-e^2)^{\frac{5}{2}}}\frac{A\cos\in\cos\alpha\sin\theta}{\pi}$$

$$and\quad \frac{1}{2\pi}\int_0^{2\pi}A\rho^3\psi_1\cos\in\sin(v-\alpha)dv = \frac{1}{2\pi}\left[\int_{\substack{-\theta\\ \Psi_1=0}}^{\theta}A\rho^3\psi_1\cos\in\sin(v-\alpha)dv + \int_{\substack{\theta\\ \Psi_1=1}}^{2\pi}A\rho^3\psi_1\cos\in\sin(v-\alpha)dv\right]$$

$$= \frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}}\frac{A\cos\in\sin\alpha\sin\theta}{\pi} \tag{3}$$

Where θ is taken to be constant

Using (3) in (1), we get

$$x''-2y'-\frac{3x}{\left(1-e^2\right)^{\frac{1}{2}}}-4Bx-\frac{\left(2+e^2\right)}{2\left(1-e^2\right)^{\frac{5}{2}}}\frac{A\cos\in\cos\alpha\sin\theta}{\pi}=-\bar{\lambda}_\alpha\left[\frac{\left(2+3e^2\right)}{2\left(1-e^2\right)^{\frac{7}{2}}}-\frac{\left(2+e^2\right)\ell_o}{2r\left(1-e^2\right)^{\frac{5}{2}}}\right]x$$

$$and \quad y''+2x'+By-\frac{\left(2+e^2\right)}{2\left(1-e^2\right)^{\frac{5}{2}}}\frac{A\cos\in\sin\alpha\sin\theta}{\pi}=-\bar{\lambda}_\alpha\left[\frac{\left(2+3e^2\right)}{2\left(1-e^2\right)^{\frac{7}{2}}}-\frac{\left(2+e^2\right)\ell_o}{2r\left(1-e^2\right)^{\frac{5}{2}}}\right]y$$

(4)

Where, $r=\sqrt{x^2+y^2}$

The condition of constraint given by (2) takes the form

$$x^2+y^2\le\left(1+\frac{e^2}{2}\right)\ell_0^2$$

(5)

From equations of motion (4) it follows that the true anomaly v does not appear explicitly in the equations of motion, so there must exist Jacobian integral for the problem.

Multiplying first and second equations of (4) by 2x’ and 2y’ respectively and adding and integrating, we get the Jacobian integral in the form.

$$x'^2+y''^2-\frac{3x^2}{\left(1-e^2\right)^{\frac{1}{2}}}-4Bx^2+By^2-\frac{\left(2+e^2\right)Ax\cos\in\cos\alpha\sin\theta}{\pi\left(1-e^2\right)^{\frac{5}{2}}}-\frac{\left(2+e^2\right)Ay\cos\in\sin\alpha\sin\theta}{\pi\left(1-e^2\right)^{\frac{5}{2}}}$$
$$+\frac{\bar{\lambda}_\alpha\left(2+3e^2\right)\left(x^2+y^2\right)}{2\left(1-e^2\right)^{\frac{7}{2}}}-\frac{\bar{\lambda}_\alpha\ell_o\left(2+e^2\right)\left(x^2+y^2\right)^{\frac{1}{2}}}{\left(1-e^2\right)^{\frac{5}{2}}}=h$$

(6)

Where, h is the constant of integration.

The curve of zero velocity is obtained by putting $x'^2+y'^2=o$ in (6) as

International Journal of Scientific & Engineering Research Volume 4, Issue 9, January-2013
ISSN 2229-5518

$$\frac{3x^2}{(1-e^2)^{\frac{1}{2}}}+4Bx^2-By^2+\frac{(2+e^2)Ax\cos\in\cos\alpha\sin\theta}{\pi(1-e^2)^{\frac{5}{2}}}+\frac{(2+e^2)Ay\cos\in\sin\alpha\sin\theta}{\pi(1-e^2)^{\frac{5}{2}}}$$
$$-\frac{\overline{\lambda}_\alpha(2+3e^2)(x^2+y^2)}{2(1-e^2)^{\frac{7}{2}}}+\frac{\overline{\lambda}_\alpha\ell_o(2+e^2)(x^2+y^2)^{\frac{1}{2}}}{(1-e^2)^{\frac{5}{2}}}+h=o \quad (7)$$

Hence we conclude that the satellite by mass m_1 will move inside the boundaries of different curves of zero velocity represented by (7) of (6) for different values of Jacobian constant h.

3.<u>Equilibrium position of the system</u>

We have obtained the system of equations (4) of the particle of mass m_1 of the system in rotating frame of reference. It has been assumed that the system is moving with effective constants and hence the string connecting the two satellites of masses m_1 and m_2 will remain always tight

The equilibrium positions of the system are given by the constant values of the coordinates in the rotating frame of reference Now, let $x = x_0$ and $y = y_0$ give the equilibrium position where x_0 and y_0 are constants.

$$\therefore x' = 0 = x'' \;\; and \;\; y' = 0 = y''$$

Thus, equations given by (4) take the following form:

$$\frac{-3x_0}{(1-e^2)^{\frac{1}{2}}}-4Bx_0-\frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}}\frac{A\cos\in\cos\alpha\sin\theta}{\pi}=-\overline{\lambda}_\alpha\left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}}-\frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}}\right]x_0$$

$$By_o-\frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}}\frac{A\cos\in\sin\alpha\sin\theta}{\pi}=-\overline{\lambda}_\alpha\left(\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}}-\frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}}\right)y_o \quad (8)$$

Where $r_0=\sqrt{{x_0}^2+{y_0}^2}$

Actually, it is impossible to find the solution of the algebraic equations (8) in its present form. Therefore, for our further investigation, it has been assumed that $\alpha=0$

This means that the sun rays is in the line of the perigee of the elliptical orbit of the centre of mass of the system. Putting $\alpha=0$, the equations of motion given by (8) take the form:

$$-\left[\frac{3}{(1-e^2)^{\frac{1}{2}}}+4B\right]x_0-\frac{(2+e^2)A_1\sin\theta}{2(1-e^2)^{\frac{5}{2}}}=-\overline{\lambda}_\alpha\left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}}-\frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}}\right]x_o$$

$$and\ By_o = -\bar{\lambda}_\alpha \left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}} \right] y_o \qquad (9)$$

From (9), we get the equilibrium points as

$$[a_1, o] = \left[\frac{(\bar{\lambda}_\alpha \ell_o + A_1 \sin\theta)\dfrac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}}}{\dfrac{\bar{\lambda}_\alpha (2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \dfrac{3}{\sqrt{1-e^2}} - 4B}, 0 \right] \qquad (10)$$

$$[b_1, c_1] = \left[\frac{-A_1 \sin\theta}{\dfrac{3}{\sqrt{1-e^2}} + 5B} \left\{ \frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}} \right\}; \sqrt{\frac{\bar{\lambda}_\alpha \ell_o (2+e^2)(1+e^2)}{\bar{\lambda}_\alpha (2+3e^2) + 2B(1-e^2)^{\frac{7}{2}}} \frac{-A_1^2 (2+e^2)\sin\theta}{4\left(\dfrac{3}{\sqrt{1-e^2}} + 5B\right)^2 (1-e^2)^5}} \right] \qquad (11)$$

Now, it can be easily seen that the equilibrium point given by (10) only gives a meaningful value of $\lambda,$ the Hook's modulus of elasticity if $0 < \theta < 90^0$

4. <u>Stability of the system</u>

We shall study the stability of the equilibrium position given by (10) of the system in the sense of Liapunov. For this, let us assume that there are small variation in the coordinate at the given equilibrium point [a_1, 0]. let η_1 and η_2 be small variation in x and y-coordinates respectively for the given position of equilibrium·

$$\therefore x = a_1 + \eta_1 \qquad \text{and} \quad y = \eta_2$$

$$\therefore x' = \eta_1' \qquad \text{and} \qquad y' = \eta_2'$$

$$x'' = \eta_2'' \qquad\qquad y'' = \eta_2'' \qquad (12)$$

Putting the values of x, y, x', y', x" and y" from (12) in (4), we get on putting and $\alpha = 0$, $\frac{A\cos\in}{\pi} = A_1$ as

$$\eta_1'' - 2\eta_2' - \left\{ \frac{3}{\sqrt{1-e^2}} + 4B \right\}(a_1 + n_1) - \frac{(2+e^2)A_1 \sin\theta}{2(1-e^2)^{\frac{5}{2}}}$$

$$= -\overline{\lambda}_\alpha \left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r(1-e^2)^{\frac{5}{2}}} \right](a_1 + \eta_1)$$

$$and \;\; \eta_2'' - 2\eta_1' + B\eta_2 = -\overline{\lambda}_\alpha \left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r(1-e^2)^{\frac{5}{2}}} \right]\eta_2 \tag{13}$$

Where, $r = \sqrt{(a_1+\eta_1)^2 + \eta_2^2}$

To obtain Jacobian integral of the equations of motion (13), we multiply first equation of (13) by 2(a₁+*n*₁)′ and second equation of (13) by η_2^2 and add them tohether, we get after integrating

$$\eta_1'^2 - 2\eta_2'^2 - \left\{ \frac{3}{\sqrt{1-e^2}} + 4B \right\}(a_1+\eta_1)^2 + B\eta_2^2 - \frac{(2+e^2)(a+\eta_1)A_1 \sin\theta}{2(1-e^2)^{\frac{5}{2}}}$$

$$+ \frac{\overline{\lambda}_\alpha (2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} \left\{(a_1+\eta_1)^2 + \mu_2^2\right\} \frac{-\overline{\lambda}_\alpha (2+e^2) l_0}{(1-e^2)^{\frac{5}{2}}} \left\{(a_1+\eta_1)^2 + \eta_2^2\right\}^{\frac{1}{2}} = h_1 \tag{14}$$

Where h_1 is the constant of integration

To test the stability in the sense of Liapunov, we take Jacobian integral (14) as Liapunov's function $V(\eta_1', \eta_2', \eta_1, \eta_2)$ and is obtained by expanding the terms of (14) as

$$V(\eta_1', \eta_2', \eta_1, \eta_2) = \eta_1'^2 + \eta_2'^2 + \eta_1^2 \left[\frac{\overline{\lambda}_\alpha (2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - 4B - \frac{3}{\sqrt{1-e^2}} \right] + \eta_2^2 \left[B + \frac{\overline{\lambda_\alpha}(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{\overline{\lambda_\alpha}\ell_0 (2+e^2)}{2a_1(1-e^2)^{\frac{5}{2}}} \right]$$

$$+ \eta_1 \left[\frac{-6a_1}{\sqrt{1-e^2}} - 8Ba_1 - \frac{(2+e^2)}{(1-e^2)^{\frac{5}{2}}} A_1 \sin\theta + \frac{\overline{\lambda_\alpha}(2+3e^2)a_1}{(1-e^2)^{\frac{7}{2}}} - \frac{\overline{\lambda_\alpha}(2+e^2)\ell_o}{(1-e^2)^{\frac{5}{2}}} \right]$$

$$+ \left[-4Ba_1^2 - \frac{3a_1^2}{\sqrt{1-e^2}} + \frac{\overline{\lambda_\alpha}(2+e^2)\ell_o a_1}{(1-e^2)^{\frac{5}{2}}} - \frac{a_1(2+e^2)A_1 \sin\theta}{(1-e^2)^{\frac{5}{2}}} \right] + 0(3) = h_1 \tag{15}$$

Where 0 (3) stand for the third and higher order terms in the small quantities η_1 *and* η_2.

Now, by Liapunov theorem on stability it follows that the only criterion for the given equilibrium position (a_1, 0) to be stable is that v defined by (15) must be positive definite and for this the following conditions must be satisfied:

(i) $$\frac{-6a_1}{(1-e^2)^{\frac{1}{2}}} - 8Ba_1 - \frac{(2+e^2)}{(1-e^2)^{\frac{5}{2}}} A_1 \sin\theta + \frac{\overline{\lambda}_\alpha (2+3e^2)a_1}{(1-e^2)^{\frac{7}{2}}} - \frac{\overline{\lambda}_\alpha (2+e^2)\ell_o}{(1-e^2)^{\frac{5}{2}}} = 0$$

(ii) $$\frac{\overline{\lambda}_\alpha (2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - 4B - \frac{3}{(1-e^2)^{\frac{1}{2}}} > 0$$

(iii) $$B + \frac{\overline{\lambda}_\alpha (2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{\overline{\lambda}_\alpha \ell_0 (2+e^2)}{2a_1(1-e^2)^{\frac{5}{2}}} > 0$$

(16)

Since [a_1, 0] given by (10) is the equilibrium point and so $a_1 > 0$ and so putting the values of a_1, it can be easily seen that all the conditions of (16) are identically satisfied.

Conclusion: Thus, we conclude that the equilibrium position [a_1, 0] of the system is stable in the sense of Liapunov.

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